Online Appendix for: Political Life Cycles

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Proofs of Formal Results

In the exposition of the game we describe the actions of the coalition as if the coalition is a unitary actor; linguistically easy, but not strictly true. There is a mass of individuals within the coalition and the coalition's action is the aggregate of their choices. To formalize this simply, suppose that the coalition's action depends upon the majority's choice; i.e., the leader is deposed if at least $W/2$ mass of supporters defect and the revolution is suppressed if at least $W/2$ mass of supporters choose to suppress. Since there is a mass of supporters, no individual is pivotal in deciding the coalition decision. By the restriction to weakly undominated strategies, every supporter's choice is their most preferred outcome. Since all coalition members are ex ante identical, all coalition members pick the same action and so the coalition members act in consort. In the main text we spoke of the coalition taking a coordinated action, and, in weakly undominated subgame perfect strategies, the equilibrium behavior of the coalition members is to all act identically.

Proof of Proposition 1: Follows directly from Bayes's Rule. ■

Since the following proofs are messy we will suppress the dependence of t and introduce the following abbreviated notation to simplify the statement of first and second order conditions. Let $u_z = u_z(g,z) = \frac{du(g,z)}{dz}$, and $u_g = u_g(g,z) = \frac{du(g,z)}{dg}$ which, by additive separability, also equals $\frac{du(g,0)}{dg}$.

Proof of Proposition 2: The proof follows from the maximization of equation 8 with respect to the leader's policy choice's g_t and z_t. We show the details for case 1. Case 2 is analogous. Equations 9 and 10 are rearrangements of the First Order Conditions: $\frac{d\mathcal{L}}{dz} = 0$ and $\frac{d\mathcal{L}}{dg} = 0$.

$$\frac{\partial \mathcal{L}}{\partial z} = -W + \Psi u_z \left(\underbrace{F'(\hat{\theta}) - (F'(\hat{\theta}) - F'(\tilde{\theta}))(H(\bar{k}) + H'(\tilde{k})\tilde{k})}_{X_1 > 0} \right)$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial g} &= -p + \Psi u_g \left(\underbrace{F'(\hat{\theta}) - (F'(\hat{\theta}) - F'(\tilde{\theta}))(H(\tilde{k}) + H'(\tilde{k})\tilde{k})}_{X_1>0} \right) \\
&+ \Psi u_g \left(\underbrace{(F(\hat{\theta}) - F(\tilde{\theta}))^2 H'(\tilde{k})}_{X_2>0} \right)
\end{aligned}
$$

where the extra term X_2 arises from the differentiation of the $(c_m - u(g,0))$ term in $\tilde{k}$. Note that the terms related to affinity do not appear in these first order conditions because they are not a function of the leader's policy choices. The second order conditions (SOC) are

$$
\begin{aligned}
\frac{\partial^2 \mathcal{L}}{\partial z^2} &= \Psi u_{zz} X_1 + \Psi u_z^2 X_3 \\
\frac{\partial^2 \mathcal{L}}{\partial g^2} &= \Psi u_{gg} \left(X_1 + X_2 \right) + \Psi u_g^2 \left(X_3 + 2X_4 + X_5 \right) < 0 \\
\frac{\partial^2 \mathcal{L}}{\partial z \partial g} &= +\Psi u_z u_g \left(X_3 + X_4 \right) < 0 \\
\frac{\partial^2 \mathcal{L}}{\partial z \partial y} &= \Psi u_z \left(X_3 \right) < 0 \\
\frac{\partial^2 \mathcal{L}}{\partial g \partial y} &= \Psi u_g \left(X_3 + X_4 \right) < 0
\end{aligned}
$$

where

$$
\begin{aligned}
X_3 &= F''(\hat{\theta}) - (F''(\hat{\theta}) - F''(\tilde{\theta})) \left(H(\tilde{k}) + H'(\tilde{k})\tilde{k} \right) \\
&- (F'(\hat{\theta}) - F'(\tilde{\theta}))^2 (c_M - u(g,0)) \left(2H'(\tilde{k}) + H''(\tilde{k})\tilde{k} \right) < 0 \\
X_4 &= (F(\hat{\theta}) - F(\tilde{\theta}))(F'(\hat{\theta}) - F'(\tilde{\theta})) \left(2H'(\tilde{k}) + \tilde{k} H''(\tilde{k}) \right) < 0 \\
X_5 &= -(F(\hat{\theta}) - F(\tilde{\theta}))^3 H''(\tilde{k})
\end{aligned}
$$

Note that if we assume H is the uniform distribution then $H'' = 0$ and so $X_5 = 0$. For F is the exponential distribution, $\Delta = F(\hat{\theta}) - F(\tilde{\theta}) = F''(\hat{\theta}) - F''(\tilde{\theta}) = -(F'(\hat{\theta}) - F'(\tilde{\theta})) > 0$ and therefore $X_4 = -2X_2$. Further, note that for the uniform distribution, $H(\tilde{k}) = H'(\tilde{k})\tilde{k}$, we can write $X_3 + 2X_1 = F'(\widehat{\theta}) - \Delta H(\tilde{k}) - \Delta H'(\tilde{k})\tilde{k} = F'(\widehat{\theta}) > 0$.

The determinant of the Jacobian is

$$
|J| = \begin{vmatrix} \frac{d^2 \mathcal{L}}{dg^2} & \frac{d^2 \mathcal{L}}{dg dz} \\ \frac{d^2 \mathcal{L}}{dg dz} & \frac{d^2 \mathcal{L}}{dz^2} \end{vmatrix}
$$

$$= \Psi^2 \left(u_{gg} u_{zz} X_1 \left(X_1 + X_2 \right) + u_{gg} u_z^2 \left(X_1 + X_2 \right) X_3 + u_g^2 u_{zz} X_1 \left(-4X_2 + X_3 \right) - 4 u_g^2 u_z^2 X_2^2 \right) > 0.$$

Given the SOC, $\frac{d^2\mathcal{L}}{dg^2} < 0$, $\frac{d^2\mathcal{L}}{dz^2} < 0$ and $|J| > 0$, for all (g, z). Hence, L's optimization problem is globally concave and the FOC characterize unique globally optimal policies. ■

For case 2, $r_W > c_W - \eta/2$, the relevant FOC that characterize optimal policies are

$$\frac{W}{u_z(g,z)\,\Psi} = \underbrace{F'(\hat{\theta}) - H(\bar{k}) \left(F'(\hat{\theta}) - F'(\bar{\theta}) \right) - \left(F(\hat{\theta}) - F(\bar{\theta}) \right) H'(\bar{k}) \left(F'(\hat{\theta}) \left(c_M - u(g_t, 0) \right) - F'(\bar{\theta}) \left(r_M - u(g_t, 0) \right) \right)}_{X_6} \quad (11)$$

and

$$\frac{p}{u_g(g,z)\,\Psi} = X_5 + \left(F(\hat{\theta}) - F(\bar{\theta}) \right)^2 H'(\bar{k}) \quad (12)$$

and proof is analogous.

Comparative Statics

Proof of Proposition 3: Via Cramer's rule,

$$\begin{aligned} \frac{dg}{dy} &= -\frac{\left| \begin{pmatrix} \frac{\partial^2\mathcal{L}}{\partial g \partial y_W} & \frac{\partial^2\mathcal{L}}{\partial g \partial z} \\ \frac{\partial^2\mathcal{L}}{\partial z \partial y_W} & \frac{\partial^2\mathcal{L}}{\partial z^2} \end{pmatrix} \right|}{|J|} \\ &= -\frac{\Psi^2 u_g \overbrace{\left(u_{zz} X_1 \left(X_3 + X_4 \right) \right)}^{+}}{|J|} \end{aligned}$$

Hence g is decreasing in y.

$$\frac{dz}{dy} = -\frac{\left| \begin{pmatrix} \frac{\partial^2\mathcal{L}}{\partial g^2} & \frac{\partial^2\mathcal{L}}{\partial z \partial y_W} \\ \frac{\partial^2\mathcal{L}}{\partial g \partial z} & \frac{\partial^2\mathcal{L}}{\partial z \partial y_W} \end{pmatrix} \right|}{|J|}$$

$$= -\frac{\Psi^2\left(\overbrace{u_{gg}u_z\left(X_1+X_2\right)X_3}^{+}+\overbrace{u_g^2u_z\left(X_3X_5-X_4^2\right)}^{-}\right)}{|J|}$$

However we cannot unambiguously sign $\frac{dz}{dy}$ because the numerator contains both positive and negative terms. Substantively these competing terms correspond to the leader cutting back on coalition rewards as y increases, while also substituting away from public goods towards a private goods focus.

However, we can show that the level of immediate coalition rewards decreases in y.

$$\begin{aligned} \frac{du(g,z)}{dy} &= \frac{-u_g\left|\begin{pmatrix} \frac{\partial^2\mathcal{L}}{\partial g\partial y_W} & \frac{\partial^2\mathcal{L}}{\partial g\partial z} \\ \frac{\partial^2\mathcal{L}}{\partial z\partial y_W} & \frac{\partial^2\mathcal{L}}{\partial z^2}\end{pmatrix}\right| - u_z\left|\begin{pmatrix} \frac{\partial^2\mathcal{L}}{\partial g^2} & \frac{\partial^2\mathcal{L}}{\partial z\partial y_W} \\ \frac{\partial^2\mathcal{L}}{\partial g\partial z} & \frac{\partial^2\mathcal{L}}{\partial z\partial y_W}\end{pmatrix}\right|}{|J|} \\ &= \frac{\Psi^2\left(-u_{gg}u_z^2\left(X_1+X_2\right)X_3-u_g^2u_{zz}X_1\left(X_3+X_4\right)+u_g^2u_z^2X_4^2\right)}{\Psi^2\left(u_{gg}u_{zz}X_1\left(X_1+X_2\right)+u_{gg}u_z^2\left(X_1+X_2\right)X_3+u_{zz}u_g^2X_1\left(X_3+2X_4\right)-u_g^2u_z^2X_4^2\right)}\end{aligned}$$

Note that every term in the numerator appears in the denominator so $\frac{du(g,z)}{dy}$ takes the form of $-\frac{Z_1}{Z_1+Z_2}$ where Z_1 and Z_2 are positive groups of terms. Hence $-1 < \frac{du(g,z)}{dy} < 0$, such that in response to increases in y the leader reduces the immediate rewards, but by less than the amount that y increases, such that the sum $u(g,z)+y$ increases in y. Since cutoffs $\hat{\theta}_t$ and $\tilde{\theta}_t$ are in linear in $u(g_t,z_t)$ and y_t, $\hat{\theta}_t$ and $\tilde{\theta}_t$ are increasing in y_t. An analogous argument shows $\tilde{k}_t$ is decreasing in y_t. ■

To proceed examine the continuation value at the beginning of period t:

$$\begin{aligned} Y_t &= \int_0^{\tilde{\theta}_t}(u(g_t,z_t)+y_t-\theta-\eta H(\tilde{k}_t))f(\theta)d\theta \\ &+ (1-H(\tilde{k}_t))\int_{\tilde{\theta}_t}^{\hat{\theta}_t}(u(g_t,z_t)+y_t-\theta)f(\theta)d\theta + H(\tilde{k}_t)(F(\hat{\theta}_t)-F(\tilde{\theta}_t))(c_W-\frac{\eta}{2}) \\ &+ (1-F(\hat{\theta}_t))c_W, \end{aligned} \tag{13}$$

where by differentiation of equation 13, $\frac{dY_t}{dy_t}\in(0,1)$.

Since $y_t = \delta(1-\rho_t)Y_{t+1}+\delta\rho\gamma$, and $\frac{dy_t}{dY_{t+1}} = \delta(1-\rho_t)$, $\frac{dy_t}{d\rho_t} = -\delta(Y_{t+1}-\gamma) < 0$. Therefore, $0 < \frac{dY_t}{dY_{t+1}} < \delta(1-\rho_t) < 1$, and $0 > \frac{dY_t}{d\rho_t} > -\delta(Y_{t+1}-\gamma)$.

Lemma 1 *For any Y_{t+1} and ρ_t there is a unique solution for Y_t.*

Proof of Lemma 1: Consider the RHS of equation 13. The RHS is increasing in Y_{t+1}. As $Y_{t+1} \to -\infty$, the coalition always deposes the leader so $RHS \to c_W$. As $Y_{t+1} \to \infty$, then coalition always stays loyal so $\frac{RHS}{Y_{t+1}} \to \delta$. Hence the RHS crosses the 45 degree line. Further, since $0 < \frac{dY_t}{dY_{t+1}} < \delta(1-\rho_t) < 1$, the crossing can occur only once. ■

Proof of Proposition 4: There is a lower bound on Y_t of c_W because the coalition could always depose the leader and obtain payoff of c_W. Via Proposition 1, as $t \to \infty$, $\rho_t \to 0$ so $y_t = \delta Y_{t+1}$ and $Y_{t+1} \to Y_t$. As $t \to \infty$, Y_t is defined recursively as the unique solution to

$$\begin{aligned}
Y_t &= \int_0^{\tilde{\theta}_t} (u(g_t, z_t) + \delta Y_{t+1} - \theta - \eta H(\tilde{k}_t)) f(\theta) d\theta \\
&+ (1 - H(\tilde{k}_t)) \int_{\tilde{\theta}_t}^{\hat{\theta}_t} (u(g_t, z_t) + \delta Y_{t+1} - \theta) f(\theta) d\theta + H(\tilde{k}_t)(F(\hat{\theta}_t) - F(\tilde{\theta}_t))(c_W - \frac{\eta}{2}) \\
&+ (1 - F(\hat{\theta}_t)) \left(c_W - H(\tilde{k}_t)\frac{\eta}{2} \right) \\
&= (u(g_t, z_t) + \delta Y_{t+1}) \left(F(\tilde{\theta}_t) H(\tilde{k}_t) + F(\hat{\theta}_t)(1 - H(\tilde{k}_t)) \right) \\
&- F(\tilde{\theta}_t) E[\theta | \theta < \tilde{\theta}_t] - (F(\hat{\theta}_t) - F(\tilde{\theta}_t))(1 - H(\tilde{k}_t)) E[\theta | \hat{\theta}_t > \theta > \tilde{\theta}_t] \\
&+ c_W \left(1 - F(\tilde{\theta}_t) H(\tilde{k}_t) - F(\hat{\theta}_t)(1 - H(\tilde{k}_t)) \right) \\
&- H(\tilde{k}_t) \eta \frac{1 - F(\tilde{\theta}_t)}{2} \\
&= \frac{1}{1 - \delta \left(F(\tilde{\theta}_t) H(\tilde{k}_t) + F(\hat{\theta}_t)(1 - H(\tilde{k}_t)) \right)} \Big[u(g_t, z_t) \left(F(\tilde{\theta}_t) H(\tilde{k}_t) + F(\hat{\theta}_t)(1 - H(\tilde{k}_t)) \right) \\
&- F(\tilde{\theta}_t) E[\theta | \theta < \tilde{\theta}_t] - (F(\hat{\theta}_t) - F(\tilde{\theta}_t))(1 - H(\tilde{k}_t)) E[\theta | \hat{\theta}_t > \theta > \tilde{\theta}_t] \\
&+ c_W \left(1 - F(\tilde{\theta}_t) H(\tilde{k}_t) - F(\hat{\theta}_t)(1 - H(\tilde{k}_t)) \right) - H(\tilde{k}_t) \eta \frac{1 - F(\tilde{\theta}_t)}{2} \Big]
\end{aligned} \tag{14}$$

Such a unique solution exists via Lemma 1. Further, for any Y_t, there exists a unique $Y_{t-1} < Y_t$. Since Y_t increases in t and ρ_t decreases in t, $y_t = \delta \rho_t \gamma + \delta(1 - \rho_t) Y_{t+1}$ is increasing in t. Via proposition 3, $\hat{\theta}_t, \tilde{\theta}_t$ and $\bar{\theta}$ increase in t and g_t and z_t decrease in t. ■

The proof of Proposition 4 did not explicitly examine how bias changes over time, a topic we now address.

Proof that ***bias*** **decreases in** t**:** $bias = \frac{p}{u_g(g,z)} / \frac{W}{u_z(g,z)} = \frac{X_1 + X_2}{X_1}$. As a simple proof we utilize the distributional functions and write $X_2 = \left(e^{-\hat{\theta}} - e^{-\tilde{\theta}}\right)^2 H'(\tilde{k})$ and writing the uniform distribution $H(\tilde{k})$ as $\tilde{k}H'(\tilde{k})$, $X_1 = e^{-\hat{\theta}} + 2\left(c_M - u(g_t, 0)\right) H'(\tilde{k}) \left(e^{-\hat{\theta}} - e^{-\tilde{\theta}}\right)^2$. As t increases, $\hat{\theta}$ and $\tilde{\theta}$ get larger and $\left(c_M - u(g_t, 0)\right)$ increases. The $\left(e^{-\hat{\theta}} - e^{-\tilde{\theta}}\right)^2$ terms get small faster than $e^{-\hat{\theta}}$, so X_2 gets smaller faster than X_1. As a result, the bias ratio moves towards 1. $\blacksquare$

Supplemental Tables

Table 4 replicates the analyses in the main text looking at alternative measures of public goods provisions. The first column replicates the Public Goods result in the main text. Columns 2-4 examine Health, Education, and National Defense Spending as a proportion of GDP using data from World Bank (2022*a*). Winning coalition size is associated with more public goods spending and transparency; however the significance is below standard levels. The negative coefficient estimates on Log(Tenure) indicate that as tenure increases leaders move in the direction of less spending on these policies, but again these results are insignificant.

Table 4: Effects of Tenure on Public Goods Measures

	Public Goods (1)	Health Spending (2)	Educ Spending (3)	Defense Spending (4)
Log(Tenure)	-0.011***	-0.030	-0.015	-0.068
	(0.003)	(0.061)	(0.051)	(0.058)
W_{t-1}	0.215***	1.47	0.031	0.119
	(0.035)	(0.956)	(0.460)	(0.686)
Log(GDPpc$_{t-1}$)	0.024*	-0.916**	0.446*	-0.476
	(0.013)	(0.375)	(0.236)	(0.310)
Log(Population$_{t-1}$)	-0.018	-1.68***	-0.007	-1.38**
	(0.024)	(0.556)	(0.452)	(0.635)
Growth	0.0005**	-0.025***	-0.023***	-0.020***
	(0.0002)	(0.008)	(0.007)	(0.007)
Resource Rents	-0.001**	-0.020**	0.007	0.013
	(0.0005)	(0.008)	(0.013)	(0.012)
Observations	6,929	3,324	4,004	5,886
R^2	0.925	0.885	0.645	0.747
Within R^2	0.182	0.046	0.017	0.029
Country fixed effects	✓	✓	✓	✓
Year fixed effects	✓	✓	✓	✓

Standard errors clustered by country

Table 5: Effects of Tenure on Equity × Spending Measures

	Health Spending × Public (1)	Educ Spending × Public (2)	Defense Spending × Public (3)
Log(Tenure)	-0.043	-0.023	-0.063**
	(0.034)	(0.035)	(0.030)
W_{t-1}	1.54***	0.751*	0.636*
	(0.516)	(0.388)	(0.366)
Log(GDPpc$_{t-1}$)	-0.857***	0.303*	-0.159
	(0.250)	(0.169)	(0.168)
Log(Population$_{t-1}$)	-1.59***	-0.151	-0.442
	(0.422)	(0.357)	(0.357)
Growth	-0.018***	-0.016***	-0.007
	(0.005)	(0.005)	(0.004)
Resource Rents	-0.015**	0.001	0.003
	(0.006)	(0.008)	(0.006)
Observations	3,324	4,003	5,879
R^2	0.946	0.804	0.783
Within R^2	0.076	0.026	0.022
Country fixed effects	✓	✓	✓
Year fixed effects	✓	✓	✓

Standard errors clustered by country
* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

	Total Rewards	Public Goods	$\frac{\text{Private}}{\text{Private + Public}}$
	(1)	(2)	(3)
Log(Tenure)	-0.019***	-0.011***	0.008***
	(0.006)	(0.003)	(0.002)
W_{t-1}	0.834***	0.215***	-0.101***
	(0.076)	(0.035)	(0.024)
Log(GDPpc$_{t-1}$)	-0.001	0.023*	-0.021**
	(0.021)	(0.013)	(0.008)
Log(Population$_{t-1}$)	-0.043	-0.019	0.025
	(0.036)	(0.024)	(0.017)
Leader Growth	-0.0003	0.0002	-0.0003
	(0.002)	(0.0007)	(0.0004)
Resource Rents	-0.001	-0.001**	0.0009***
	(0.0010)	(0.0005)	(0.0003)
Observations	6,929	6,929	6,929
R^2	0.860	0.925	0.941
Within R^2	0.349	0.180	0.124
Country fixed effects	✓	✓	✓
Year fixed effects	✓	✓	✓

Standard errors clustered by country
* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$

Table 6: Alternative Explanations: Leader Growth (Competence)